CHAPTER 15

Jarrow Turnbull Derivatives, 2000

INTEREST RATE DERIVATIVES

15.0 INTRODUCTION

When we price equity derivatives, the initial value of the stock is given, and we construct a lattice of future stock prices. An analogous procedure works for pricing **interest rate derivatives**. Consider pricing an option written on a Treasury bill. Given the initial price of the Treasury bill, we must model its possible values over the life of the option. We must do this in a way that (1) is consistent with the absence of arbitrage, (2) is consistent with the initial term structure, and (3) recognizes that the Treasury bill pays a known fixed amount (the principal) at maturity.

Describing this arbitrage-free evolution of the Treasury bill's price is equivalent to modeling the evolution of the term structure of interest rates. This presents a more difficult problem than that encountered for equity derivatives. A number of different solutions exist for this problem. We choose to model the evolution of the term structure by concentrating on the short-term interest rate.

In modeling the short-term interest rate, it is essential to specify how many sources of uncertainty affect its evolution. For this chapter we assume that there is only one source of uncertainty, giving a **one-factor model**. If we assumed that there are two sources of uncertainty affecting the evolution of the interest rate, this would create a **two-factor model**.[1] Using a one-factor model is a strong assumption because the empirical evidence suggests that there is more than one factor. Nonetheless, we choose the one-factor model to illustrate the procedure. Once this case is mastered, the multiple-factor extension follows similarly in a straightforward fashion.

15.1 CONSTRUCTION OF THE LATTICE

Here we explain how to construct a lattice of future spot interest rates. We use the binomial model of Chapter 4, but this time for interest rate movements. In the next section, we will formalize our discussion about the underlying assumptions.

[1] The difference between a one- and two-factor model can be understood as follows. A one-factor model has one source of uncertainty. The randomness underlying its evolution at any node can be generated via tossing *one* (unbiased) coin. A two-factor model has two sources of uncertainty. The generation of the evolution of the term structure at any node necessitates the tossing of at least *two* (unbiased) coins.

For pricing interest rate derivatives, we must construct a lattice of spot interest rates that is consistent with the observed initial term structure. For clarity, we choose the time between changes in the spot rate of interest to be one year. In practice, a shorter interval would be used, depending on the degree of accuracy required.

The spot rate of interest corresponds to the rate of interest over the interval in the lattice. Because our interval length is one year, we are modeling the one-year spot interest rate. If our interval length had been one week, we would be modeling the one-week spot interest rate.

To illustrate the construction, let us consider the initial term structure in Table 15.1. In this table, the first column gives the years until maturity. The second column gives the zero-coupon bond prices. The third column gives the yield-to-maturity for each bond. The last column refers to the volatility of the spot rate of interest. In the last column, the first number, 0.017, refers to the volatility of the spot interest rate at the end of the first year. The second number, 0.015, refers to the volatility of the spot interest rate at the end of the second year, given that the spot interest rate at the end of the first year is known. This specification of the volatility at the different future intervals is referred to as the **term structure of volatilities.**

Let $B(0,t)$ denote the date-0 value of a zero-coupon default-free bond that matures at the end of year t. From Table 15.1, observe that $B(0,1) = 0.9399$. For convenience, we use continuously compounded interest rates. The current one-year rate of interest, $r(0)$, is defined by

$$0.9399 = \exp[-r(0)],$$

implying

$$r(0) = 6.1982 \text{ percent.}$$

TABLE 15.1 *Interest Rates Are Assumed to Be Normally Distributed*

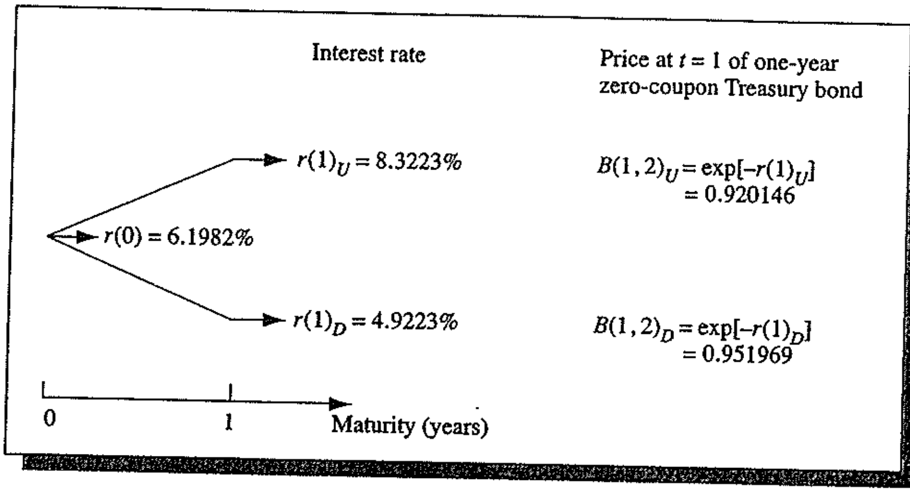
INITIAL DATA			
MATURITY (YEARS)	BOND PRICES* $B(0,T)$	YIELD** (PERCENT)	VOLATILITY***
1	0.9399	6.1982	0.017
2	0.8798	6.4030	0.015
3	0.8137	6.8721	0.011
4	0.7552	7.0193	0.0075

*All bonds have zero coupons and are default-free.

**Continuously compounded yield.

***Volatility refers to the volatility of the spot interest rate.

Figure 15.1 *Finding the Short-Term Rates to Price a Two-Year Zero-Coupon Treasury Bond*



At the end of the first year, we assume that the one-year spot interest rate can take one of only two possible values, denoted by $r(1)_U$ and $r(1)_D$, respectively. This is the one-factor assumption. We must pick these values to be consistent with the initial term structure as shown in Table 15.1; we do this by trial and error. Let us guess the values

$$r(1)_U = 8.3223 \text{ percent}$$

and

$$r(1)_D = 4.9223 \text{ percent}$$

as shown in Figure 15.1.

Let $B(1,2)$ denote the value at date 1 of a zero-coupon default-free bond that matures at date 2. As shown in Figure 15.1, using the date-1 spot interest rates, we can compute the possible date-1 bond prices as

$$B(1,2) = \begin{cases} 0.920146 & \text{if} \quad r(1)_U \text{ occurs.} \\ 0.951969 & \text{if} \quad r(1)_D \text{ occurs.} \end{cases}$$

In building this lattice, it is essential to construct it so that there are no arbitrage opportunities implicitly within it. We showed in Chapter 6 that in a binomial lattice for equity derivatives, there is no arbitrage between the stock and the money market account if and only if there exists a unique probability such that the stock price normalized by the money market account's value follows a martingale.

We referred to this probability as a martingale probability. Fortunately, this theorem can be generalized to apply to the term structure of zero-coupon default-free bond prices as well.

Although we do not prove this result here, we use the insight. Analogously stated, in a one-factor model, there is no arbitrage among all the zero-coupon bonds and the money market account if and only if there exists a unique probability such that all the zero-coupon bond prices normalized by the money market account's value follow a martingale.[2] We utilize the "if" part of this theorem for the subsequent analysis.

We want to ensure that the evolution of the term structure of interest rates is arbitrage-free. That is, we want to ensure that no arbitrage opportunities exist among the zero-coupon bonds and the money market account. To do this via the above-stated theorem, we assume that there exists a unique martingale probability such that normalized zero-coupon bond prices follow a martingale.

This condition is that

$$B(0,2)/A(0) = E^{\pi}[B(1,2)/A(1)], \tag{15.1}$$

where $A(t)$ denotes the value of the money market account at date t.

Without loss of generality, we assume that the martingale probability of each state occurring is 0.5.[3]

By definition, $A(0) \equiv 1$ and $A(1) = \exp(r(0)) = \exp(0.061982)$. Substituting into Expression (15.1) gives

$$\begin{aligned} B(0,2) &= \exp(-0.061982)(0.5 \times 0.920146 + 0.5 \times 0.951969) \\ &= 0.8798, \end{aligned}$$

which agrees with the value in Table 15.1. Our guess on the possible values for the spot interest rate at date 1 is correct.

We must also check that our estimates are consistent with the volatility given in Table 15.1. The volatility of the spot interest rate at time t is defined to be the standard deviation of the change in the spot interest rate over the next time interval. In symbols,

$$\sigma(t) \equiv \sqrt{\text{var}^{\pi}(\Delta r(t))} \equiv \sqrt{E^{\pi}\{[\Delta r(t) - E^{\pi}(\Delta r(t))]^2\}}.$$

[2]For a proof of this result, see Jarrow (1995).

[3]This assumption is without loss of generality because in constructing a lattice there are usually three unknowns: (i) the magnitude of the spot rate "up," (ii) the magnitude of the spot rate "down," and (iii) the martingale probability. We usually have two constraints: (i) the expected change in the spot rate, and (ii) the volatility of the spot rate. Three unknowns-and two constraints gives 1 degree of freedom, and this degree of freedom allows the specification of $\pi = \frac{1}{2}$.

The volatility is computed using the martingale probabilities. Volatility measures the dispersion or the variability in spot rates across time. As such, it is an important statistic for understanding the term structure of the interest rate's evolution. The volatility is determined[4] by

$$\sigma(0) = [r(1)_U - r(1)_D]/2. \tag{15.2}$$

By convention, $r(1)_U$ is greater than $r(1)_D$.

Substituting for $r(1)_U$ and $r(1)_D$ gives

$$\begin{aligned}[r(1)_U - r(1)_D]/2 &= (0.083223 - 0.049223)/2 \\ &= 0.017,\end{aligned}$$

which agrees with Table 15.1. Had we been wrong, we would have revised $r(1)_U$ and $r(1)_D$ by iteration until these two conditions were satisfied.

Continuing, at the end of the first year, if the one-year rate of interest is $r(1)_U$, then at the end of the second year the one-year spot rate can have one of two possible values: $r(2)_{UU}$ or $r(2)_{UD}$. Similarly, if the one-year rate of interest at date 1 is $r(1)_D$, then the spot rate at date 2 can take the values $r(2)_{DU}$ or $r(2)_{DD}$, shown in Figure 15.2. The lattice is constructed so that it recombines.

We need to choose $r(2)_{UU}$, $r(2)_{DU}$, and $r(2)_{DD}$ to be consistent with the observed term structure of interest rates and volatilities, as shown in Table 15.1.

We first match the volatilities. Let us guess that $r(2)_{UU} = 10.8583$ percent. From Table 15.1, the spot rate volatility is 0.015, implying that for $r(2)_{UU}$ greater than $r(2)_{UD}$,

$$\sigma(1) = 0.015 = [r(2)_{UU} - r(2)_{UD}]/2, \tag{15.3}$$

which in turn implies that

$$\begin{aligned}r(2)_{UD} &= 0.108583 - 2 \times 0.015 \\ &= 7.8583 \text{ percent.}\end{aligned}$$

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[4] If a random variable can take one of two possible values, a or b, with probability p and $(1 - p)$, respectively, then

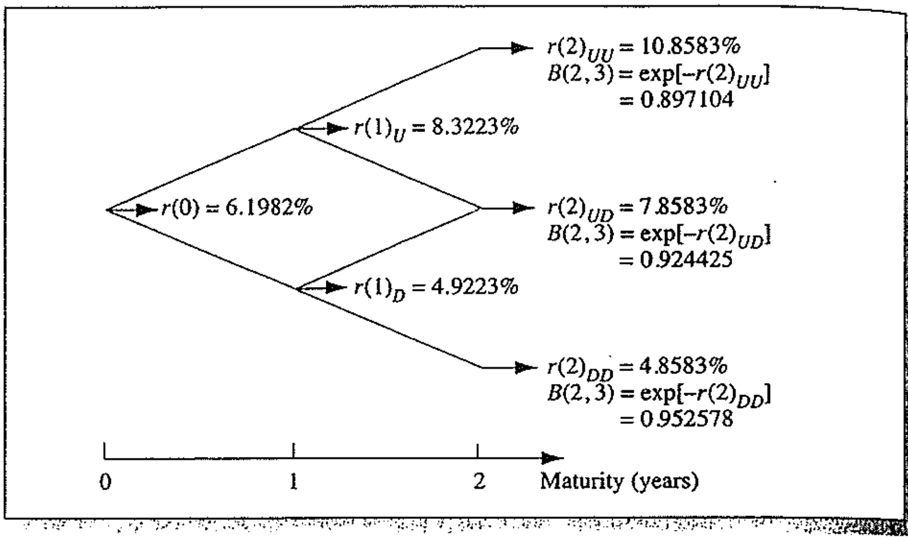
1) expected value $= a \times p + b \times (1 - p)$

and

2) variance, σ^2
$= \{a - [ap + b(1-p)]\}^2 p + \{b - [ap + b(1-p)]\}^2 (1-p)$
$= (a - b)^2 p(1 - p)$.

In this example, $p \equiv 1/2$, $a \equiv r(1)_U$, and $b \equiv r(1)_D$.

FIGURE 15.2 *Finding the Short-Term Rates to Price a Three-Year Zero-Coupon Treasury Bond*



If at date 1 the one-year spot interest rate is $r(1)_D$, then because $r(2)_{DU}$ is greater than $r(2)_{DD}$,

$$\sigma(1) = 0.015 = [r(2)_{DU} - r(2)_{DD}]/2,$$

implying

$$\begin{aligned} r(2)_{DD} &= 0.078583 - 2 \times 0.015 \\ &= 4.8583 \text{ percent.} \end{aligned}$$

These choices of the date-2 spot rates match the term structure of volatilities. We next check to see if these values are consistent with the initial term structure of interest rates.

Using the three different spot rates previously computed, the three values for the bond prices for $B(2,3)$ are shown in Figure 15.2.

To ensure no arbitrage, the bond prices in the lattice must satisfy the following condition:

$$B(1,3)/A(1) = E^{\pi}[B(2,3)/A(2)]. \tag{15.4}$$

We compute these values. At the end of the first year, the spot interest rate can either be 8.3223 percent or 4.9223 percent. Suppose that we are at the upper node in the lattice, where $r(1)_U = 8.3223$ percent. The value of the money market account at the end of the first year is

$$A(1) = \exp(0.061982), \tag{15.5}$$

and at the end of the second year it is

$$A(2)_U = \exp(r(0)) \times \exp(r(1)_U) = \exp(0.061982) \times \exp(0.083223). \tag{15.6}$$

It is important to realize that at the end of the first year, given that we are at the upper node, this value of $A(2)_U$ is known. Therefore, substituting Expressions (15.5) and (15.6) into Expression (15.4) and simplifying gives

$$\begin{aligned} B(1,3)_U &\equiv \exp(-0.083223)E^{\pi}[B(2,3)] \\ &= \exp(-0.083223) \times (0.5 \times 0.897104 + 0.5 \times 0.924425) \\ &= 0.838036. \end{aligned}$$

Figure 15.3 *Lattice of Short-Term Rates*

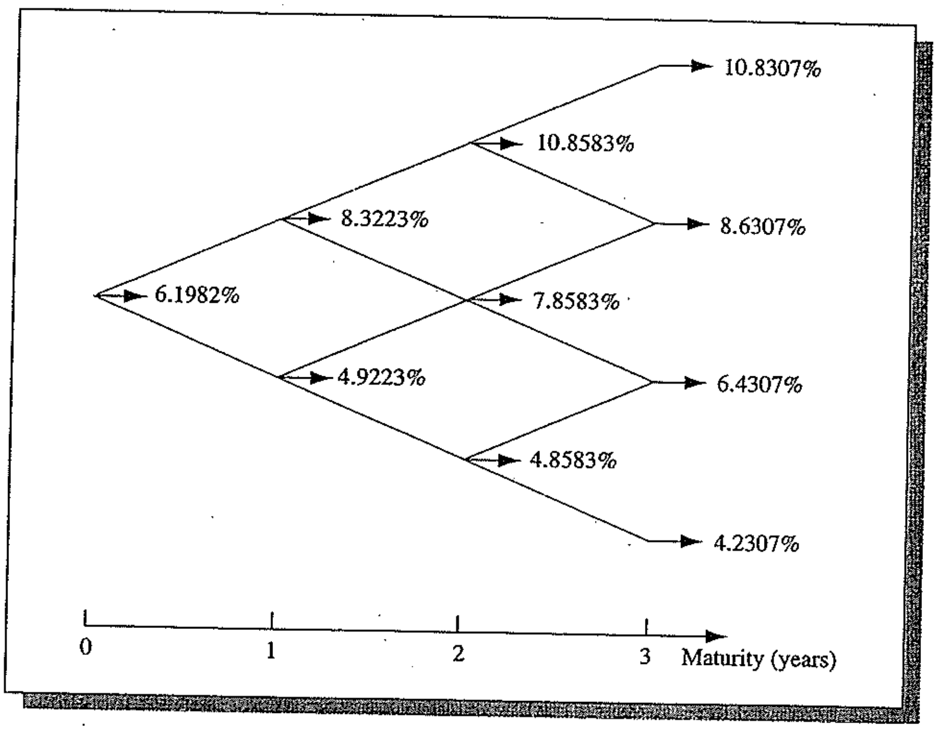
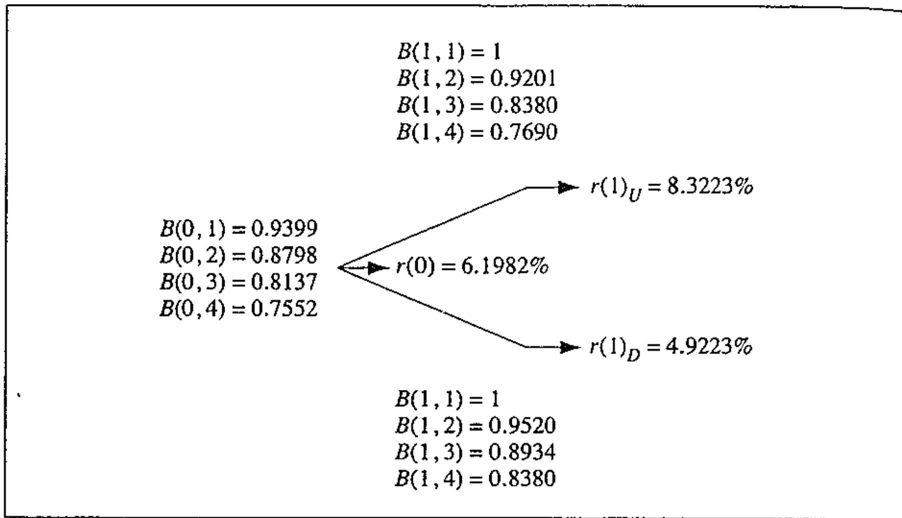


FIGURE 15.4 *Evolution of the Term Structure*



Suppose that at the end of the first year we are at the lower node in the lattice, so the spot interest rate $r(1)_D$ is 4.9223 percent. The value of the money market account at the end of the first year is

$$A(1) = \exp(0.061982)$$

and at the end of the second year it is

$$A(2)_D = \exp(r(0)) \times \exp(r(1)_D) = \exp(0.061982) \times \exp(0.049223).$$

Again, $A(2)_D$ is known at this node. Substituting into Expression (15.4) and simplifying gives

$$\begin{aligned} B(1,3)_D &= \exp(-0.049223) \times (0.5 \times 0.924425 + 0.5 \times 0.952578) \\ &= 0.893424. \end{aligned}$$

Finally, at time 0, to ensure no arbitrage, the price of $B(0,3)$ must satisfy the following condition:

$$B(0,3)/A(0) = E^{\pi}[B(1,3)/A(1)].$$

We check to see if the above condition is satisfied. Substituting in the relevant quantities gives

$$
\begin{aligned}
B(0,3) &= \exp(-0.061982) \times (0.5 \times B(1,2)_U + 0.5 \times B(1,2)_D) \\
&= \exp(-0.061982) \times (0.5 \times 0.838036 + 0.5 \times 0.893424) \\
&= 0.8137,
\end{aligned}
$$

which agrees with the initial value in Table 15.1.

We leave to the reader to verify that the spot interest rates given in Figure 15.3 correctly price the four-year zero-coupon Treasury bill in Table 15.1.

Figure 15.3 describes the spot interest rate process given the initial term structure of interest rates and volatilities. It also implies the evolution of the term structure of interest rates, illustrated in Figure 15.4, in which the initial term structure is shown and its evolution is given up to date 1.

Our discussion of lattice construction is now complete.

15.2 SPOT RATE PROCESS

Here we present the general model for the spot rate process. We formalize our previous example and explain the underlying structure. For the most part, this involves little more than replacing numbers with algebraic symbols.

First we need to make some assumptions about the probability distribution approximated by the evolution of the spot interest rates in our lattice. We consider two possible cases: (1) changes in interest rates follow a normal probability distribution, and (2) changes in interest rates follow a lognormal probability distribution. Both cases are used in practice.

Normal Distribution

If one believes that changes in spot interest rates are normally distributed,[5] then the evolution of the spot interest rate can be described by

$$\Delta r(t) = [a(t) - b(t)r(t)]\Delta + \sigma(t)\Delta W(t), \tag{15.7}$$

where $\Delta r(t) \equiv r(t + \Delta) - r(\Delta)$, Δ is the length of the time interval, $a(t)$, $b(t)$ are parameters, $\sigma(t)$ is the volatility at date t, and $\Delta W(t)$ is a normally distributed random variable with zero mean and variance Δ.

Expression (15.7) is under the equivalent martingale probabilities and implies that spot interest rates are normally distributed. This normality assumption is a continuous time limit of the Ho-Lee (1986) model.

The parameters $a(t)$, $b(t)$, and $\sigma(t)$ are deterministic functions of time and are independent of the spot rate, $r(t)$. This functional dependence on date t is necessary for the implied zero-coupon bond prices to be calibrated to match the observed initial term structure of interest rates.

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[5] As discussed in the following text, this belief is under the martingale probabilities, *not* the actual or empirical probabilities.

Under Expression (15.7) we have

$$E^{\pi}(\Delta r(t)) = [a(t) - b(t)r(t)]\Delta$$

and

$$\sqrt{\text{var}}^{\pi}(\Delta r(t)) = \sigma(t)\sqrt{\Delta}.$$

The expected change in spot rates is determined by the parameters $a(t)$ and $b(t)$. The volatility of the change in the spot rate is $\sigma(t)\sqrt{\Delta}$. These parameters can be estimated from historic time series observations of changes in spot rates using standard statistical procedures. Alternatively, they can be implicitly estimated by calibrating the model's values for bonds and derivatives to market prices.

Analogously to Chapter 4, we can approximate this process with a binomial model:

$$r(t+\Delta) - r(t) = \begin{cases} [a(t) - b(t)r(t)]\Delta + \sigma(t)\sqrt{\Delta} & \text{with probability } 1/2 \\ [a(t) - b(t)r(t)]\Delta - \sigma(t)\sqrt{\Delta} & \text{with probability } 1/2 \end{cases}. \tag{15.8}$$

For example, if today's spot interest rate is $r(0)$, at the end of the first interval the spot interest rate takes one of two values:

$$r(1)_U \equiv r(0) + [a(0) - b(0)r(0)]\Delta + \sigma(0)\sqrt{\Delta} \quad \text{with probability } 1/2$$

and

$$r(1)_D \equiv r(0) + [a(0) - b(0)r(0)]\Delta - \sigma(0)\sqrt{\Delta} \quad \text{with probability } 1/2. \tag{15.9}$$

Note that $r(1)_U > r(1)_D$, given that the volatility $\sigma(0)$ is positive.

From Expression (15.9) we have

$$[r(1)_U - r(1)_D]/2 = \sigma(0)\sqrt{\Delta}. \tag{15.10}$$

We can now recognize Expression (15.2) in the previous example as being a special case of Expression (15.10) in which the time interval Δ is set equal to one.

It is easy to generalize Expression (15.10). At date t, if the spot interest is $r(t)$, then one period later, using Expression (15.8), the interest rate will be either

$$r(t+\Delta)_U - r(t) = [a(t) - b(t)r(t)]\Delta + \sigma(t)\sqrt{\Delta}$$

or

$$r(t+\Delta)_D - r(t) = [a(t) - b(t)r(t)]\Delta - \sigma(t)\sqrt{\Delta}.$$

Subtracting the above two equations gives

$$[r(t+\Delta)_U - r(t+\Delta)_D]/2 = \sigma(t)\sqrt{\Delta}. \tag{15.11}$$

This expression shows how the term structure of volatilities determines the *spread* between the two possible spot interest rates at date $t + \Delta$.

The *level* of the spot interest rates at date $t + \Delta$ is determined by the initial term structure of interest rates via the martingale relation satisfied by the normalized zero-coupon bond prices.

The path of the short-term interest rate over two intervals is shown in Figure 15.5. Referring to Figure 15.5, we investigate when the lattice recombines, that is, whether $r(2)_{UD}$ equals $r(2)_{DU}$.

For these two rates to be equal, we require that

$$\begin{aligned} &r(1)_U + [a(1) - b(1)r(1)_U]\Delta - \sigma(1)\sqrt{\Delta} \\ &= r(1)_D + [a(1) - b(1)r(1)_D]\Delta + \sigma(1)\sqrt{\Delta}, \end{aligned}$$

which implies

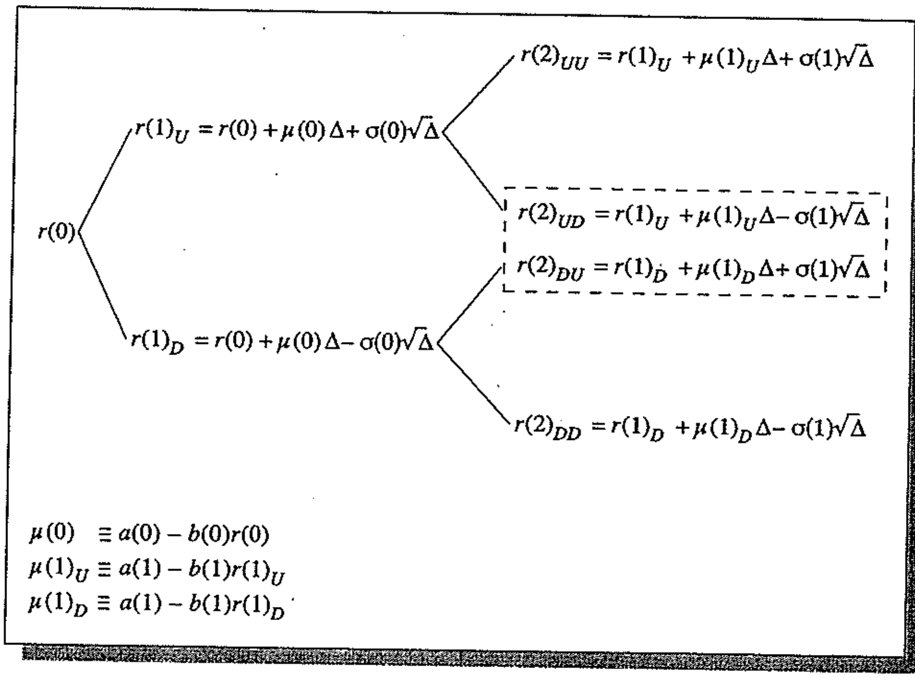
$$[r(1)_U - r(1)_D]/2 - b(1)\Delta[r(1)_U - r(1)_D]/2 = \sigma(1)\sqrt{\Delta}.$$

Substituting Expression (15.10) into this equation gives, after simplification,

$$[1 - b(1)\Delta]\sigma(0) = \sigma(1). \tag{15.12}$$

This equation determines the parameter $b(1)$.

FIGURE 15.5 *The Short-Term Rate Process*



Expression (15.12) implies that the parameters $b(1)$, $\sigma(0)$, and $\sigma(1)$ are related. In Expression (15.12), for $b(1)$ to be positive, that is, $b(1) > 0$, the volatility of the spot rate at date 1, $\sigma(1)$ must be less than the volatility at date 0, $\sigma(0)$, that is, the volatility of the spot rate of interest must decrease over time. If $b(1) = 0$, volatilities are constant across time. Finally, if $b(1) < 0$, volatilties increase.

We can show that if $b(t)$ is positive, that is, $b(t) > 0$, the time-conditional variance of $r(t)$ is bounded. This follows because $\sigma(t)$ is decreasing in t, so its largest value (the upper bound) occurs at date 0. It is not the case if $b(t)$ is zero for all t, for then the variance is proportional to t. The Ho-Lee (1986) model, for example, assumes $b(t)$ is zero.

In the financial economics literature, Expression (15.7) is usually thought of as implying **mean reversion**, that is, spot interest rates tend toward a long-run mean. This is misleading because Expression (15.7) describes the interest rate movements under the equivalent martingale probabilities and not the empirical or true probabilities. The behavior of spot rates differ under these two different probabilities. An alternative and more precise way of describing Expression (15.7) is to say that the unconditional variance of the spot interest rate is finite.[6]

Given that we have determined the spot rate at each node of the lattice, we do not need to explicitly calculate the actual values of the parameters $\{a(t)\}$ and $\{b(t)\}$. They are implied by the lattice. It should be noted that the term structure of volatilities implicitly determines the parameter $\{b(t)\}$, while the term structure of interest rates implicitly determines the parameter $\{a(t)\}$.

EXAMPLE

Normal Distribution

We illustrate here the calculation of the parameters $a(t)$, $b(t)$ for normally distributed spot rates. We use the information in Figure 15.3 and Table 15.1 to compute the values of $a(t)$ and $b(t)$. The length of each interval is one year, implying $\Delta = 1$.

Recall that Figure 15.3 gives the spot interest rate evolution consistent with the initial term structure of interest rates and volatilities in Table 15.1. For the first period, using Expression (15.9),

$$\begin{aligned} a(0) - b(0)r(0) &= r(1)_U - r(0) - \sigma(0) \\ &= 0.083223 - 0.061982 - 0.017 \\ &= 0.004241 \end{aligned}$$

and

$$\begin{aligned} a(0) - b(0)r(0) &= r(1)_D - r(0) + \sigma(0) \\ &= 0.049223 - 0.061982 + 0.017 \\ &= 0.004241. \end{aligned}$$

[6] This is explained in Musiela *et al.* (1993).

This is one equation in two unknowns. Without additional information we do not have an unique solution, so arbitrarily[7] we set

$$a(0) = 0.004241$$
$$b(0) = 0.0.$$

For the second period, if $r(1)_U = 8.3223$ percent,

$$\begin{aligned} a(1) - b(1) \times r(1)_U &= r(2)_{UU} - r(1)_U - \sigma(1) \\ &= 0.108583 - 0.083223 - 0.015 \\ &= 0.01036 \end{aligned}$$

and

$$\begin{aligned} a(1) - b(1) \times r(1)_U &= r(2)_{UD} - r(1)_U + \sigma(1) \\ &= 0.078583 - 0.083223 + 0.015 \\ &= 0.01036. \end{aligned}$$

From Expression (15.12),

$$\begin{aligned} b(1) &= 1 - \sigma(1)/\sigma(0) \\ &= 1 - 0.015/0.017 \\ &= 0.117647. \end{aligned}$$

Therefore,

$$a(1) = 0.02015094.$$

If the spot interest is $r(1)_D = 4.9223$ percent, we need to check that we get the same values for $a(1)$ and $b(1)$. Similar computations yield

$$\begin{aligned} a(1) - b(1) \times r(1)_D &= r(2)_{DU} - r(1)_D - \sigma(1) \\ &= 0.078583 - 0.049223 - 0.015 \\ &= 0.01436 \end{aligned}$$

and

$$\begin{aligned} a(1) - b(1) \times r(1)_D &= r(2)_{DD} - r(1)_D + \sigma(1) \\ &= 0.048583 - 0.049223 + 0.015 \\ &= 0.01436. \end{aligned}$$

[7] Rather than arbitrarily determining $a(0)$, $b(0)$, we could have calibrated them to some other market observable.

Given the value of $b(1)$, then

$$a(1) = 0.01436 + b(1) \times 0.049223$$
$$= 0.02015094,$$

implying that we have consistency.

We leave it to the reader to verify that

$$a(2) = 0.01767947$$

and

$$b(2) = 0.2666667.$$

■

Lognormal Distribution

In the previous section we studied a model in which spot interest rates are normally distributed. This is a convenient assumption, for it allows us to derive closed form solutions for many types of interest rate derivatives. However, this assumption implies that spot interest rates can be negative. We can see this via Expression (15.7) by noticing that if $\Delta W(t)$ takes on a large negative value (by chance), then $r(t + \Delta)$ can be negative. Negative spot rates generate zero-coupon bond prices above their face value. This situation is inconsistent with the availability of cash currency, which can be stored at no cost.[8] For this reason, it is usually considered an undesirable property.

One way to avoid this implication is to assume that the logarithm of the spot interest rate is normally distributed. Let

$$\nu(t) \equiv \ln [r(t)].$$

Assuming that $\nu(t)$ is normally distributed implies that the interest rate $r(t)$ cannot be negative and is lognormally distributed.[9]

Based on this insight, we assume that

$$\Delta\nu(t) = [a(t) - b(t)\nu(t)]\Delta + \sigma(t)\Delta W(t), \tag{15.13}$$

where $\Delta\nu(t) \equiv \nu(t + \Delta) - \nu(t) \equiv \ln r(t + \Delta) - \ln r(\Delta)$. This assumption underlies the Black, Derman, and Toy (1990) model.

[8]Surprisingly, in November of 1998, Japanese yen deposits in Western banks paid negative interest rates. News reports attributed this to credit risk, an issue we discuss in Chapter 18.

[9]For example, given that $r(t) = \exp[\nu(t)]$, if $\nu(t) = -2.5$, then $r(t) = 0.082$, which is positive.

Under Expression (15.13) we have

$$E^{\pi}(\Delta v(t)) = [a(t) - b(t)v(t)]\Delta$$

and

$$\sqrt{\text{var}}^{\pi}(\Delta v(t)) = \sigma(t)\sqrt{\Delta}.$$

The expected change in the $\ln r(t)$ is determined by the parameters $a(t)$ and $b(t)$. The volatility of the change in $\ln r(t)$ is $\sigma(t)\sqrt{\Delta}$. These parameters can be estimated historically using time series data or implicitly using market prices.

Because Expression (15.13) is similar to Expression (15.7), the binomial lattice of spot interest rates approximating Expression (15.13) can be constructed in a similar manner.

We illustrate this construction now. Given the spot interest rate at date t, $r(t)$, we compute $v(t) = \ln r(t)$. Using Expression (15.13), the value of v at the next interval is

$$v(t+\Delta) - v(t) = \begin{cases} [a(t) - b(t)v(t)]\Delta + \sigma(t)\sqrt{\Delta} & \text{with probability } 1/2 \\ [a(t) - b(t)v(t)]\Delta - \sigma(t)\sqrt{\Delta} & \text{with probability } 1/2. \end{cases} \tag{15.14}$$

In the "up state"

$$v(t+\Delta)_U - v(t) = [a(t) - b(t)v(t)]\Delta + \sigma(t)\sqrt{\Delta}$$

and in the "down state"

$$v(t+\Delta)_D - v(t) = [a(t) - b(t)v(t)]\Delta - \sigma(t)\sqrt{\Delta}.$$

Subtracting the above two equations gives

$$[v(t+\Delta)_U - v(t+\Delta)_D]/2 = \sigma(t)\sqrt{\Delta}, \tag{15.15}$$

which is analogous to Expression (15.11). In terms of the spot interest rate, this becomes

$$\{\ln[r(t+\Delta)_U / r(t+\Delta)_D]\}/2 = \sigma(t)\sqrt{\Delta}. \tag{15.16}$$

Compare Expression (15.11) with Expression (15.16). The difference in these expressions implies that the magnitude of the volatilities for the lognormal distribution can be substantially greater than those for the normal distribution. In

TABLE 15.2 *Interest Rates Are Assumed to Be Lognormally Distributed*

INITIAL DATA			
MATURITY (YEARS)	BOND PRICES* $B(0,T)$	YIELD (PERCENT)	VOLATILITY**
1	0.9399	6.1982	0.2
2	0.8798	6.4030	0.18
3	0.8137	6.8721	0.17

*All bonds have zero coupons and are default-free.

**Volatility refers to the volatility of the logarithm of the spot interest rate.

Table 15.2, a term structure of volatilities is shown, assuming that interest rates are lognormally distributed. Compare the magnitude of the numbers in this table to those in Table 15.1. The numbers in Table 15.1 are approximately equal to $r(t)$ times the numbers in Table 15.2. This makes sense, as $\sqrt{\text{var}}(\Delta r(t)) = \sigma(t)\sqrt{\Delta}$ in Table 15.1, using Expression (15.7), and $\sqrt{\text{var}}\,(\Delta r(t)) = r(t)\sigma(t)\,\sqrt{\Delta}$ in Table 15.2, using Expression (15.13).[10]

[10]Using Expression (15.13),

$$\text{var}[\Delta v(t)] = \sigma(t)^2\Delta.$$

By definition,

$$\Delta v(t) \equiv \ln r(t+\Delta) - \ln r(t).$$

We can write

$$\begin{aligned} r(t+\Delta) &\equiv r(t) + \Delta r(t) \\ &= r(t)[1 + \Delta r(t)/r(t)] \end{aligned}$$

so that

$$\ln r(t+h) = \ln r(t) + \ln[1 + \Delta r(t)/r(t)].$$

Hence

$$\begin{aligned} \Delta v(t) &= \ln[1 + \Delta r(t)/r(t)] \\ &\simeq \Delta r(t)/r(t). \end{aligned}$$

Therefore,

$$\text{var}[\Delta v(t)] = \{\text{var}[\Delta r(t)]\}/r(t)^2,$$

implying

$$\sqrt{\text{var}}[\Delta r(t)] = r(t)\sigma(t)\sqrt{\Delta}.$$

'ly Distributed

	VOLATILITY**
	0.2
	0.18
	0.17

e.

; that interest rates are
mbers in this table to
imately equal to $r(t)$
$(\Delta r(t)) = \sigma(t)\sqrt{\Delta}$ in
$\cdot(t)\sigma(t)\sqrt{\Delta}$ in Table

EXAMPLE

Lognormal Distribution

We illustrate in this example the computation of the parameters for lognormally distributed spot rates. We use the information in Table 15.2 to construct the lattice of spot interest rates for the lognormal case. The current spot interest rate is given by

$$0.9399 = \exp[-r(0)],$$

implying

$$r(0) = 6.1982 \text{ percent}.$$

At date 1, let us guess that

$$r(1)_U = 7.9221 \text{ percent, which implies that } B(1,2)_U = 0.923836,$$

and

$$r(1)_D = 5.3103 \text{ percent, which implies that } B(1,2)_D = 0.948282.$$

The current value of the two-year zero-coupon bond, using Expression (15.1), is

$$\begin{aligned} B(0,2) &= \exp(-0.061982) \times [0.923836 \times (1/2) + 0.948282 \times (1/2)] \\ &= 0.8798. \end{aligned}$$

This agrees with the value in Table 15.2.

We must also check that our estimates are consistent with the volatilities given in Table 15.2.

$$y(1)_U = \ln[r(1)_U] = \ln(0.079221)$$

and

$$y(1)_D = \ln[r(1)_D] = \ln(0.053103).$$

Substituting into Expression (15.15) gives

$$\begin{aligned} \sigma(0) &= [\ln(0.079221) - \ln(0.053103)]/2 \\ &= 0.20, \end{aligned}$$

which agrees with Table 15.2. This completes the spot rate determination at date 1.

At date 2, let us guess that

$r(2)_{UU} = 10.8922$ percent, which implies that $B(2,3)_{UU} = 0.896800$,
$r(2)_{UD} = 7.5993$ percent, which implies that $B(2,3)_{UD} = 0.926823$,

and

$r(2)_{DD} = 5.3018$ percent, which implies that $B(2,3)_{DD} = 0.948363$.

Substituting into Expression (15.15) gives the volatility for the upper node in the lattice:

$$\begin{aligned}\sigma(1) &= [\ln(0.108922) - \ln(0.075993)]/2\\ &= 0.18\end{aligned}$$

and, for the lower node in the lattice,

$$\begin{aligned}\sigma(1) &= [\ln(0.075993) - \ln(0.053018)]/2\\ &= 0.18.\end{aligned}$$

These estimates match the volatility specification in Table 15.2.

At $t = 1$,

$$\begin{aligned}B(1,3)_U &= \exp(-0.079221) \times [0.896800 \times (^1/_2) + 0.926823 \times (^1/_2)]\\ &= 0.842364\end{aligned}$$

and

$$\begin{aligned}B(1,3)_D &= \exp(-0.053103) \times [0.926823 \times (^1/_2) + 0.948363 \times (^1/_2)]\\ &= 0.889103.\end{aligned}$$

The current value of the three-year zero-coupon bond is

$$\begin{aligned}B(0,3) &= \exp(-0.061982) \times [0.842364 \times (^1/_2) + 0.889103 \times (^1/_2)]\\ &= 0.8137,\end{aligned}$$

which agrees with Table 15.2. The binomial lattice is shown in Figure 15.6.

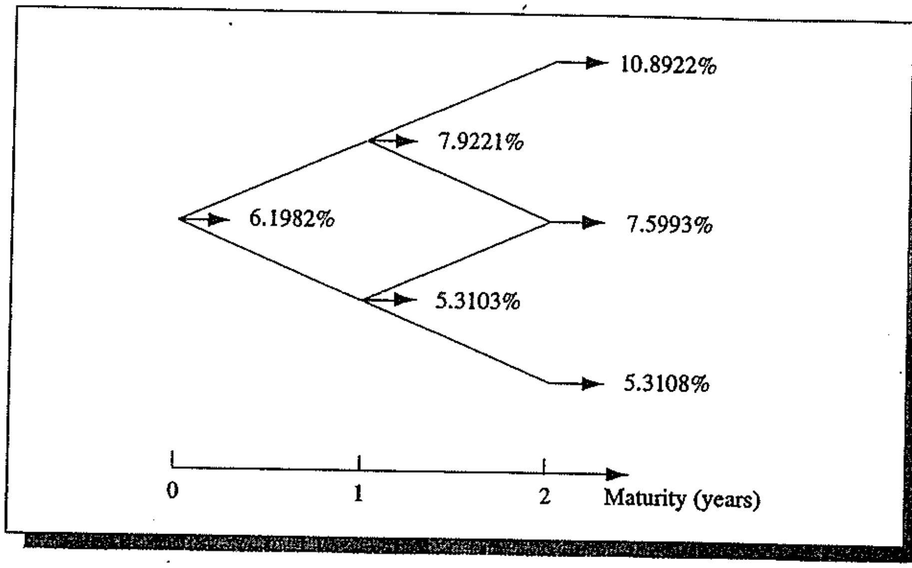
We leave it to the reader to verify that over the first period, the values of the parameters for the spot interest rate process are given by

$$a(0) = 0.0454$$

and

$$b(0) = 0.0.$$

Figure 15.6 *Short-Term Interest Rates, Lognormal Distribution*



For the second period,

$$a(1) = -0.1152$$

and

$$b(1) = 0.1.$$ ■

15.3 VALUING OPTIONS ON TREASURY BILLS

Given the arbitrage-free evolution of the spot interest rate process, the risk-neutral valuation procedure of Chapter 6 enables us to price interest rate derivatives. In this section we show how to use the material in Chapter 6 to price and hedge options on Treasury bills.

Consider a European call option that matures at date T, with strike price K, written on a Treasury bill that matures at a later date, T_1. The value of the call option at expiration is

$$c(T) \equiv \begin{cases} B(T,T_1) - K & \text{if} \quad B(T,T_1) > K \\ 0 & \text{if} \quad B(T,T_1) \le K. \end{cases} \tag{15.17}$$

The value of a put option with the same expiration date, strike price, and written the same Treasury bill is

$$p(T) \equiv \begin{cases} 0 & \text{if} \quad B(T,T_1) \geq K \\ K - B(T,T_1) & \text{if} \quad B(T,T_1) < K. \end{cases} \qquad (15.1$$

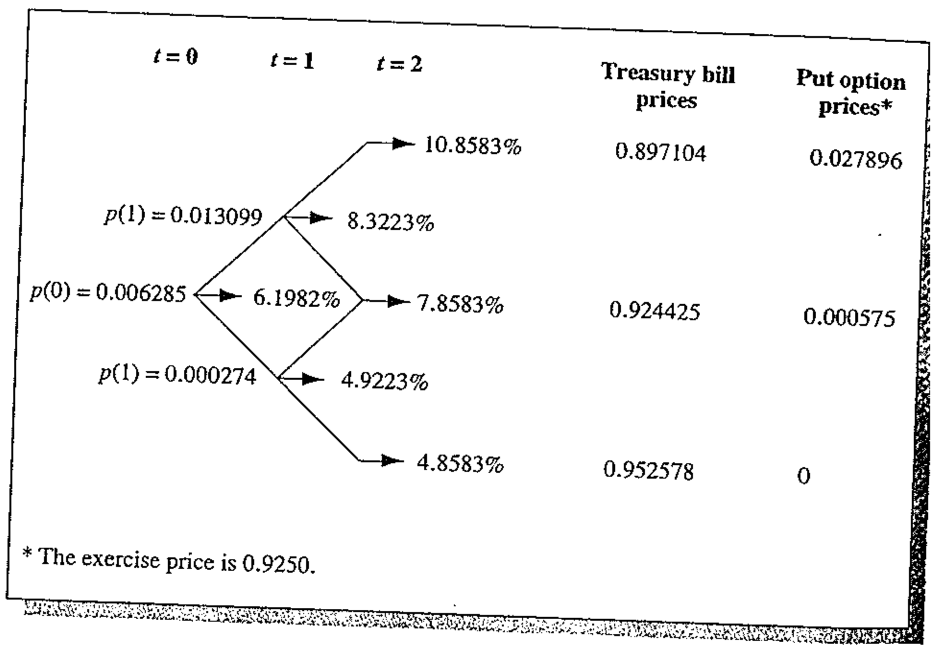
Given an initial term structure of interest rates and volatilities, the first step valuing these options is to construct an arbitrage-free lattice of short-term intere rates consistent with these volatilities. This procedure was illustrated in the previo sections. The second step is to use the constructed lattice and the risk-neutral valu tion procedure of Chapter 6. We illustrate this procedure here.

Put Options

Here we study put option valuation using the risk-neutral valuation procedure o Chapter 6. Consider a two-year European put option written on a Treasury bill with maturity of one year when the option expires. Let the strike price be 0.925.

We use the lattice of short-term interest rates shown in Figure 15.2 and repro duced in Figure 15.7. This lattice is based on the initial term structure of volatilitie

FIGURE 15.7 *Pricing a European Put Option on Treasury Bills (Based on Figure 15.2)*



given in Table 15.1. We observe in Figure 15.7 that there are three possible Treasury bill prices at date 2 and thus three possible put option prices (0.027896, 0.000575, 0).

At the end of the first year, if the spot interest rate is 8.3223 percent, the value of the put option is

$$\begin{aligned} p(1)_U &= \exp(-0.083223)E^{\pi}[p(2)] \\ &= \exp(-0.083223) \times (0.5 \times 0.027896 + 0.5 \times 0.000575) \\ &= 0.013099 \end{aligned}$$

and, if the spot rate is 4.9223 percent, the value of the put option is

$$\begin{aligned} p(1)_D &= \exp(-0.049223) \times (0.5 \times 0.000575 + 0.5 \times 0) \\ &= 0.000274. \end{aligned}$$

The value of the put option today is

$$\begin{aligned} p(0) &= \exp(-0.061982)E^{\pi}[p(1)] \\ &= \exp(-0.061982) \times (0.5 \times 0.013099 + 0.5 \times 0.000274) \\ &= 0.006285. \end{aligned}$$

This completes the valuation procedure.

A Replicating Portfolio

Suppose we want to construct a portfolio to replicate this option. From Table 15.1, we have the one-, two-, three-, and four-year zero-coupon bonds at our disposal. How many assets do we need to replicate the option? At date $t = 1$, the end of the first interval, the option takes one of two possible (different) values, either 0.013099 or 0.000274. Therefore, we need at least two different assets in the replicating portfolio. But which two assets should we use? We could use any pair of the bonds—either

a) the one-year and two-year zero-coupon bonds,
b) the one-year and three-year zero-coupon bonds,

or

c) the two-year and three-year zero-coupon bonds.

For simplicity we ignore the four-year zero-coupon bond, but it could also be included. We could use the money market account as one of our assets, but it would be equivalent to rolling over the one-year zero-coupon bond in the replicating portfolio and is therefore omitted.

Consider using the one-year and two-year zero-coupon bonds to form the synthetic put. At date 0, the value of this replicating portfolio is

$$V(0) = n_1\, 0.9399 + n_2\, 0.8798,$$

where n_1 is the number of one year zero-coupon bonds and n_2 the number of two-year zero-coupon bonds.

By construction, the value of the replicating portfolio at the end of the first period must equal the value of the put option. Referring to Figures 15.1 and 15.7, this implies that

$$n_1 1 + n_2\, 0.920146 = 0.013099$$

and

$$n_1 1 + n_2\, 0.951969 = 0.000274,$$

which gives two equations in two unknowns. The solution is

$$n_1 = 0.383927$$

and

$$n_2 = -0.40301. \tag{15.19}$$

The cost of constructing the replicating portfolio gives the value of the synthetic put option, that is,

$$\begin{aligned} V(0) &= n_1\, 0.9399 + n_2\, 0.8798 \\ &= 0.006285, \end{aligned}$$

which must be the value of the traded put option to avoid arbitrage.

If we use one-year and three-year zero-coupon bonds to form the replicating portfolio, the portfolio holdings can be shown to be

$$n_1 = 0.207145$$

and

$$n_3 = -0.231548, \tag{15.20}$$

where n_1 is the number of one-year zero-coupon bonds and n_3 the number of three-year zero-coupon bonds. The cost of constructing the replicating portfolio is

$$\begin{aligned} V(0) &= n_1 \times 0.9399 + n_3 \times 0.8137 \\ &= 0.006285, \end{aligned}$$

which represents the arbitrage-free value of the traded put. Of course, this is the same value computed earlier.

Finally, if we use two-year and three-year zero-coupon bonds to form the replicating portfolio, then

$$n_2 = 0.472227$$

and

$$n_3 = -0.502965, \tag{15.21}$$

where n_2 is the number of two-year zero-coupon bonds and n_3 the number of three-year zero-coupon bonds. The cost of constructing the synthetic put is

$$\begin{aligned} V(0) &= n_2 \times 0.8798 + n_3 \times 0.8137 \\ &= 0.006285, \end{aligned}$$

which is the value of the put option.

Each portfolio, by construction, replicates the payoff to the put option. From a theoretical viewpoint we should be indifferent in choosing among the three different replicating portfolios. In practice, however, other considerations may indicate a preference for a particular replicating portfolio. We will return to this topic in the next chapter.

Call Options

The valuation procedure for European call options on Treasury bills is identical to that used for European puts. To gauge your understanding, you should check the values given in Figure 15.8 for a two-year European call option with strike price 0.925 written on a zero-coupon bond that has a maturity of one year when the option expires.

Put-Call Parity

Here we discuss put-call parity between options on Treasury bills. Consider a European call option and a European put option, both written on a Treasury bill that matures at date T_1. The options expire at date T and have a strike price of K.

The put-call parity relationship between the options is

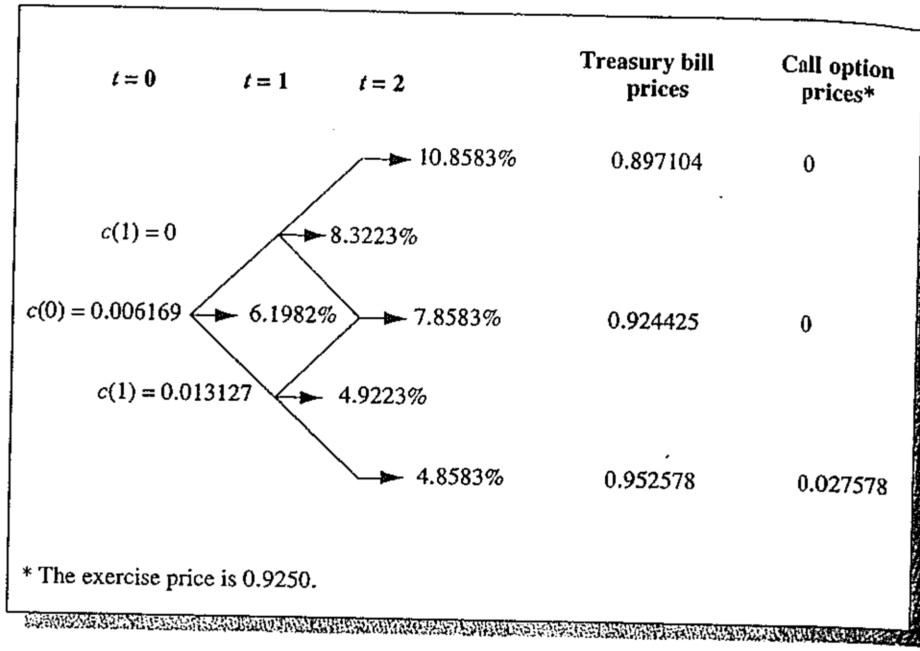
$$p(t) + B(t, T_1) = c(t) + K \times B(t, T). \tag{15.22}$$

The proof of this put-call parity relation is identical to that contained in Chapter 3 and is therefore omitted.

Let us use the results in Figures 15.7 and 15.8 and Table 15.1 to verify Expression (15.22). From Figure 15.7 and Table 15.1, the left side of Expression (15.22) is

$$\begin{aligned} p(0) &= 0.0063 \\ B(0,3) &= 0.8137 \\ \text{Total} &= 0.82. \end{aligned}$$

FIGURE 15.8 *Pricing a European Call Option on Treasury Bills (Based on Figure 15.2)*



From Figure 15.8 and Table 15.2, the right side of Expression (15.22) is

$$c(0) = 0.006169$$
$$K \times B(0,2) = 0.925 \times 0.8798$$
$$\text{Total} = 0.82.$$

These values are the same, which verifies Expression (15.22).

15.4 TREASURY BILL FUTURES

We now show how to price and hedge using Treasury bill futures. In practice, futures contracts are often more liquid securities than the Treasury bills themselves, making them a better hedging instrument.

Given the arbitrage-free lattice of short-term interest rates, the determination of futures prices is relatively straightforward.

Pricing

Let us demonstrate how to determine Treasury bill futures prices. Consider a futures contract that is written on a one-year Treasury bill. Let the futures contract delivery date be at the end of the second year. Let $\mathcal{F}(t,2)$ denote the futures price of this contract at date $t = 0, 1, 2$.

At the delivery date of the contract, the futures price equals the spot price of the one-year Treasury bill. Hence

$$\mathcal{F}(2,2) = B(2,3).$$

In Figure 15.9, we see that three values are possible: 0.897104, 0.924425, and 0.952578.

At the end of the first year, two spot interest rates are possible: 8.3223 percent and 4.9223 percent. Suppose that we are at the upper node with a spot interest rate of 8.3223 percent.

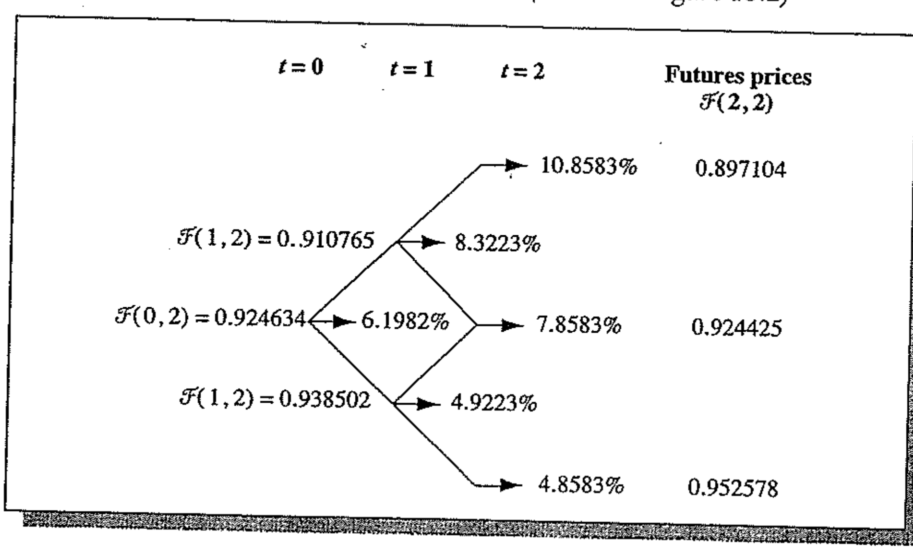
Under the martingale probabilities, we know from Chapter 6 that futures prices are martingales. Therefore, from Expression (6.25) of Chapter 6,

$$\begin{aligned}\mathcal{F}(1,2)_U &= E^{\pi}[\mathcal{F}(2,2)]\\ &= 0.5 \times 0.897104 + 0.5 \times 0.924425\\ &= 0.910765.\end{aligned}$$

If we are at the lower node so that the spot interest rate is 4.9223 percent, the futures price is similarly determined:

$$\begin{aligned}\mathcal{F}(1,2)_D &= E^{\pi}[\mathcal{F}(2,2)]\\ &= 0.5 \times 0.924425 + 0.5 \times 0.952578\\ &= 0.938502.\end{aligned}$$

FIGURE 15.9 *Treasury Bill Futures Prices (Based on Figure 15.2)*



Today, the futures price is

$$\begin{aligned}\mathcal{F}(0,2) &= E^{\pi}[\mathcal{F}(1,2)] \\ &= 0.5 \times 0.910765 + 0.5 \times 0.938502 \\ &= 0.924634.\end{aligned}$$

This completes the calculation of the Treasury bill futures prices.

Hedging

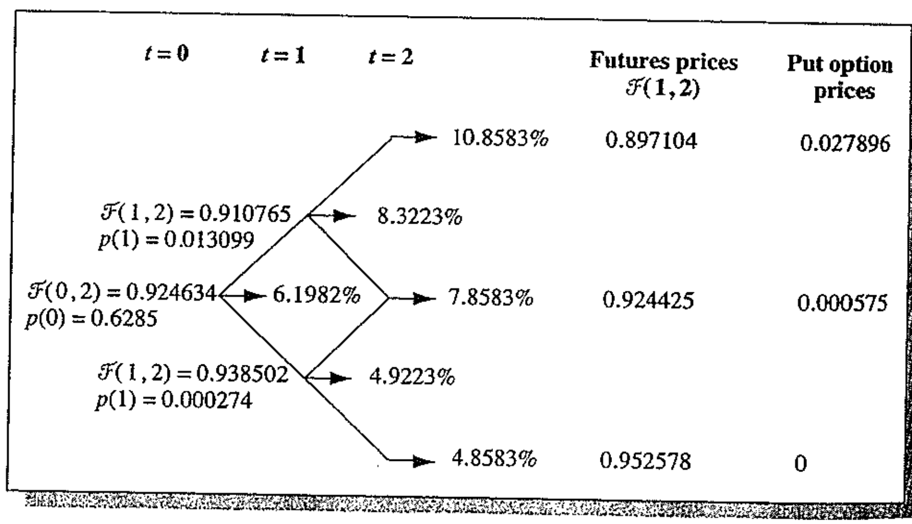
We now study hedging with Treasury bill futures. We have derived futures prices (Figure 15.9) and put option prices (Figure 15.7) for contracts written on one-year Treasury bills. These results are summarized in Figure 15.10.

We define the put option's delta with respect to this futures contract in the same way as we define deltas for equity options. It is the ratio of the change in the option prices across the two possible states to the change in the futures prices. For the put option, the delta is

$$\begin{aligned}\Delta_p &= (1.3099 - 0.0274)/(0.910765 - 0.938502) \\ &= -0.4624.\end{aligned}$$

Suppose that we want to form a portfolio using the futures contract to replicate the option. Today we invest B dollars in the short-term riskless asset and enter into m

FIGURE 15.10 *Treasury Bill Futures Prices and Put Option Prices (Based on Figures 15.7 and 15.9)*



futures contracts.[11] Each dollar invested in the short-term riskless asset yields 1.0639 ($= \exp(0.061982)$). The initial cost of constructing the portfolio is

$$V(0) = m \times 0 + B,$$

given that the initial value of the futures contract is zero.

If the short-term interest rate goes to 8.3223 percent, the option value is 0.013099. By construction, we need

$$0.013099 = m(0.910765 - 0.924634) + B(1.0639),$$

where the first term on the right side is the cash flow from the investment in the futures contracts.

If the spot interest rate goes to 4.9223 percent, the option value is 0.000274. By construction, we need

$$0.000274 = m(0.938502 - 0.924634) + B(1.0639).$$

This gives two equations in two unknowns. Solving for m and B gives

$$m = (0.013099 - 0.000274)/(0.910765 - 0.938502)$$
$$= -0.4624$$

and

$$B = 0.6285.$$

The solution is to short 0.4624 futures contracts and invest 0.6285 dollars in the riskless asset.

Given that this portfolio replicates the payoffs to the put option, the value of the traded option must equal the cost of constructing this synthetic option:

$$p(0) = V(0) = B.$$

If we had written the traded put option as well, this synthetic put held in conjunction with the traded option would yield a hedged portfolio. Indeed, by writing 0.4624 futures contracts and investing 0.6285 dollars in the riskless asset, we can completely offset the risk of writing the option. This would be a zero-investment position because the investment in the short-term riskless asset is financed by the proceeds from writing the option.

Our discussion of Treasury bill futures contracts is now complete.

[11] This is equivalent to forming the portfolio using n_1 one-year Treasury bills and m futures contracts.

15.5 SUMMARY

We show in this chapter how to construct an arbitrage-free lattice of spot interest rates that is consistent with the following: (1) the current term structure of interest rates and (2) the current term structure of volatilities. In constructing this lattice, we must specify the process describing the evolution of the spot interest rate. We consider two distributions; the first has interest rates normally distributed, which implies that

$$[r(t + \Delta)_U - r(t + \Delta)_D]/2 = \sigma(t)\sqrt{\Delta}.$$

The second distribution has interest rates that are lognormally distributed, which implies that

$$\{\ln[r(t + \Delta)_U/r(t + \Delta)_D]\}/2 = \sigma(t)\sqrt{\Delta}.$$

Once we construct the lattice of spot interest rates using the risk-neutral valuation procedure of Chapter 6, it is possible to price interest rate derivatives such as options on Treasury bills, Treasury bonds, and interest rate futures contracts.

We have also shown how to hedge interest rate derivatives with other types of interest rate derivatives. For example, to hedge an option written on a Treasury bill, we showed how to use either other Treasury bills or Treasury bill futures contracts. The lattice approach can also be applied to (1) price American type options, where it may be optimal to prematurely exercise the option, and (2) price interest rate exotics to which early exercise or boundary conditions are attached.

As we mentioned in the introduction to this chapter, there are a number of different approaches available for pricing interest rate derivatives in an arbitrage-free manner. The approach we use here is to model the spot interest rate process. An alternative approach is to model the evolution of the entire forward rate curve. This alternative approach is identified with Heath, Jarrow, and Morton (1992). A discrete time description of this model can be found in Jarrow (1995). In Chapter 16 we study the Heath-Jarrow-Morton approach in the context of a continuous trading model.

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QUESTIONS

Question 1 Put-Call Parity for European Treasury Bill Options

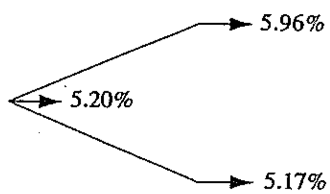
Consider a call option and a put option that mature at date T and with strike price K. The options are written on a Treasury bill that matures at date T_1. Prove the following:

$$p(t;T) + B(t,T_1) = c(t;T) + K \times B(t,T).$$

(Hint: The proof is very similar to the put-call parity result in Chapter 3.)

Question 2

In the following figure, you are given the lattice of six-month continuously compounded interest rates, which have been derived using the lognormal spot rate model. The time interval used in the lattice is six months.



The equivalent martingale probability of an up or down state occurring is $\frac{1}{2}$. The payoff to an interest rate cap that matures in six months' time is defined by

$$\text{cap}(0) \equiv \text{Max}\left\{\frac{[R(6) - K](1/2)}{1 + R(6)(1/2)}, 0\right\} \times \text{Principal},$$

where $R(6)$ is the six-month (simple) interest rate when the option matures, K is the strike price, and Principal is the principal amount.

a) The interest rates in the lattice are continuously compounded rates, while the payoff to the cap is defined in terms of simple interest rates. Show that the payoff to the cap can be written in the form